
Tags:: #on/Quantum-Computing , #on/Quantum-Computing/Linear-Algebra, #on/Linear-Algebra
Links:: 📝 Introduction to Classical and Quantum Computing - Thomas Wong

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Linear Algebra is to Quantum Computing as Boolean Algebra is to Classical Computing. Although we have to learn a new tool, it makes calculations much easier.

Quantum States

Column Vectors

- We write $|0\rangle$ and $|1\rangle$ as column vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- It is easier to write superpositions this way. A generic qubit would be:

$$\begin{aligned} |\psi\rangle &= \alpha|0\rangle + \beta|1\rangle \\ &= \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \end{aligned}$$

Row Vectors

- The transpose of a Column vector is obtained by rewriting it as a row vector, and it is denoted by ^{T}.

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}^T = (\alpha \quad \beta).$$

- In quantum computing, we typically use the conjugate transpose, which is obtained by taking the complex conjugate of each component of the transpose. It is denoted by †.

$$\begin{pmatrix}\alpha \\ \beta\end{pmatrix}^\dagger = (\alpha^* \quad \beta^*).$$

- A bra $\langle\psi|$ is the conjugate transpose of a ket, and conversely, a ket is the conjugate transpose of a bra.

$$\langle\psi| = |\psi\rangle^\dagger, \qquad |\psi\rangle = \langle\psi|^\dagger.$$

Inner Products

Inner Products Are Scalars

- The inner product of $|\psi\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix}\gamma \\ \delta\end{pmatrix}$ is defined as $\langle\psi|\phi\rangle$, which is called a *bra-ket* or *bracket*:

$$\langle\phi|\psi\rangle = (\alpha^* \quad \beta^*)\begin{pmatrix}\gamma \\ \delta\end{pmatrix}$$
$$\langle\phi|\psi\rangle = \alpha^*\gamma + \beta^*\delta.$$

  which is a scalar value. That's why an inner product is also known as scalar product.
- The inner product of $|\psi\rangle$ and $|p\rangle hi$ is just the complex conjugate of the inner product of $|\psi\rangle$ and $|\phi\rangle$:

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*.$$

Orthonormality

- The inner product of $|\psi\rangle$ with itself, denoted $\langle\psi|\psi\rangle$, is just the total probability, i.e. $|\alpha|^2 + |\beta|^2$, and if it is 1, then the state $|\psi\rangle$ is *normalized*.
- Any two states on opposite sides of the Bloch sphere have zero inner product, and the states with zero inner product are called *orthogonal* states.
- *Orthonormal* states are those states that are both *normalized* and *orthogonal* to each other. e.g. $\{|0\rangle, |1\rangle\}$ are orthonormal, so are

$\{|+\rangle, |-\rangle\}$, and $\{|i\rangle, |-i\rangle\}$.

Projection, Measurement, and Change of Basis

- For an orthonormal basis $\{|a\rangle, |b\rangle\}$, the state of a qubit can be written as

$$|\psi\rangle = \alpha|a\rangle + \beta|b\rangle,$$

  where $\alpha = \langle a|\psi\rangle$ and $\beta = \langle b|\psi\rangle$. Here $\langle a|\psi\rangle$ is the amplitude of $|\psi\rangle$ in $|a\rangle$, i.e. the amount of $|\psi\rangle$ that is in $|a\rangle$, or the amount of overlap between $|\psi\rangle$ and $|a\rangle$, which in mathematical terms is called the projection of $|\psi\rangle$ onto $|a\rangle$.
- Inner products can be used to find the amplitudes and orthonormality of a state in a certain basis, which provides a convenient way to change basis states, and perform calculations using different computer algebra systems.

Quantum Gates

Gates as Matrices

- Quantum gates are matrices that keep the total probability equal to 1.

Common One-Qubit Gates as Matrices

- The previously introduced common One-Qubit Quantum Gates can be represented as matrices:

Gate	Action on Computational Basis	Matrix Representation
Identity	$I\|0\rangle = \|0\rangle$ $I\|1\rangle = \|1\rangle$	$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Pauli X	$X\|0\rangle = \|1\rangle$ $X\|1\rangle = \|0\rangle$	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Pauli Y	$Y\|0\rangle = i\|1\rangle$ $Y\|1\rangle = -i\|0\rangle$	$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
Pauli Z	$Z\|0\rangle = \|0\rangle$ $Z\|1\rangle = -\|1\rangle$	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Phase S	$S\|0\rangle = \|0\rangle$ $S\|1\rangle = i\|1\rangle$	$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
T	$T\|0\rangle = \|0\rangle$ $T\|1\rangle = e^{i\pi/4}\|1\rangle$	$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$
Hadamard H	$H\|0\rangle = \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$ $H\|1\rangle = \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Chapter 3 - Linear Algebra#Common One-Qubit Gates as Matrices

CHAPTER 3 - LINEAR ALGEBRA_COMMON_ONE_QUBIT_GATES_MATRICES.EXCALIDRAW.SVG

Sequential Quantum Gates

- Using linear algebra, we can compute the effect of a sequence of quantum gates.
- For example: $HSTH|0\rangle$ can be computed by simplifying/multiplying the matrices together:

$$HSTH|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Circuit Identities

- We can prove different circuit identities using Linear Algebra, e.g. $HXH = Z$ can be proven by multiplying the matrices together, either manually or, in most cases, using a computer program.

Unitarity

- Quantum gates are unitary matrices, and unitary matrices are quantum gates.

Reversibility

- A quantum gate is always reversible, and its inverse is $U^\dagger$.

Outer Products

Outer Products Are Matrices

- As opposed to inner products $\langle\psi|\phi\rangle$ which are scalar, an outer product $|\psi\rangle\langle\phi|$ is always a matrix:

$$|\psi\rangle\langle\phi| = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \gamma^* & \delta^* \end{pmatrix}.$$

- We can add outer products together to construct various quantum gates.
- The outer product of $|\phi\rangle$ and $|\psi\rangle$ is just the conjugate transpose of the outer product of $|\psi\rangle$ and $|\phi\rangle$:

$$|\phi\rangle\langle\psi| = |\psi\rangle\langle\phi|^\dagger.$$

Completeness Relation

- A complete orthonormal basis $\{|a\rangle, |b\rangle\}$ satisfies the completeness relation

$$|a\rangle\langle a| + |b\rangle\langle b| = I.$$

Summary

- The mathematical language of quantum computing is linear algebra.
- Quantum states are represented by column vectors called kets, and the conjugate transpose of a ket is a bra.

- Multiplying a bra and a ket is an inner product that yields the projection or amplitudes of the states onto each other.
- States with zero inner product are orthogonal, and a state whose inner product with itself is 1 is normalized.
- All quantum gates are unitary matrices.
- A quantum gate is always reversible.
- Multiplying a ket and a bra is an outer product, which is a matrix.
- A complete orthonormal basis satisfies the completeness relation, meaning the sum of the outer products of each basis vector with itself equals the Identity matrix.